MATH2010D Advanced Calculus I 2019-20

§1 Vectors in $\mathbb{R}^n$

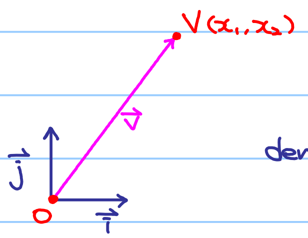
$\mathbb{R}^n$ and Vector Operations

Definition 1.1

A vector in $\mathbb{R}^n$ is an element of $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_1, x_2, \cdots, x_n \in \mathbb{R}\}$.

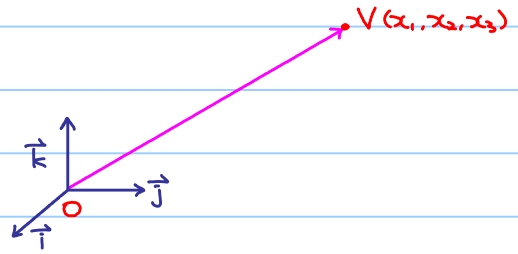
Example 1.1

A vector in $\mathbb{R}^2$ can be written as (x_1, x_2) or $x_1\vec{i} + x_2\vec{j}$.



denoted by $\vec{v}$ or $\overrightarrow{OV}$

A vector in $\mathbb{R}^3$ can be written as (x_1, x_2, x_3) or $x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$.



A vector in $\mathbb{R}^n$ can be written as $(x_1, x_2, \cdots, x_n)$ or $x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$.

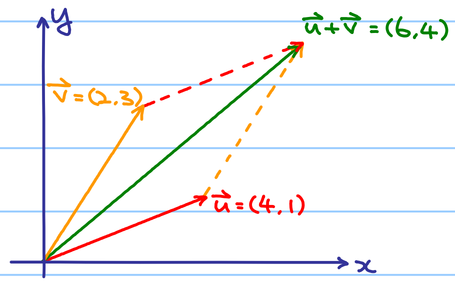
$\vec{0} = (0, 0, \cdots, 0) = 0\vec{e}_1 + 0\vec{e}_2 + \cdots + 0\vec{e}_n$ is said to the zero vector.

Definition 1.2 (Vector Addition)

If $\vec{u} = (u_1, u_2, \cdots, u_n)$, $\vec{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$, $\vec{u} + \vec{v} = (u_1+v_1, u_2+v_2, \cdots, u_n+v_n)$.

Example 1.2

If $\vec{u} = (4, 1)$, $\vec{v} = (2, 3) \in \mathbb{R}^2$

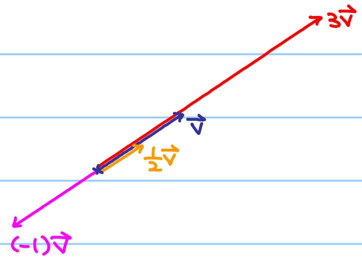


Definition 1.3 (Scalar Multiplication)

If $\vec{v} = (v_1, v_2, \cdots, v_n)$, $t \in \mathbb{R}$ (called scalar), $t\vec{v} = (tv_1, tv_2, \cdots, tv_n)$.

Example 1.3

If $\vec{v} = (4,2) \in \mathbb{R}^2$, $3\vec{v} = (12,6)$, $\frac{1}{2}\vec{v} = (2,1)$, $(-1)\vec{v} = (-4,-2)$.



Definition 1.4

$\vec{v}, \vec{w} \in \mathbb{R}^n$ are said to be parallel if $\vec{v} = t\vec{w}$ for some $t \in \mathbb{R}$.

Definition 1.5

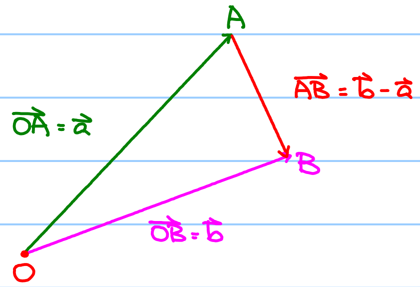
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$.

$-\vec{v}$ is defined as $(-1)\vec{v}$ and $\vec{u} - \vec{v}$ is defined as $\vec{u} + (-\vec{v})$.

Example 1.4

If $\overrightarrow{OA} = \vec{a} = 3\vec{i} + 5\vec{j}$ and $\overrightarrow{OB} = \vec{b} = 4\vec{i} + 2\vec{j}$,

then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a} = (4\vec{i} + 2\vec{j}) - (3\vec{i} + 5\vec{j}) = \vec{i} - 3\vec{j}$.



Proposition 1.1

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $s, t \in \mathbb{R}$.

1) (Commutative Law of Vector Addition) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2) (Associative Law of Vector Addition) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3) (Existence of Additive Identity) $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$
4) (Existence of Additive Inverse) $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$
5) (Existence of Multiplicative Identity) $1\vec{v} = \vec{v}$ where $1 \in \mathbb{R}$
6) (Associative Law of Scalar Multiplication) $s(t\vec{v}) = (st)\vec{v}$
7) (Distributive Law of Scalar Multiplication) $s(\vec{u} + \vec{v}) = s\vec{u} + s\vec{v}$ and $(s+t)\vec{v} = s\vec{v} + t\vec{v}$.

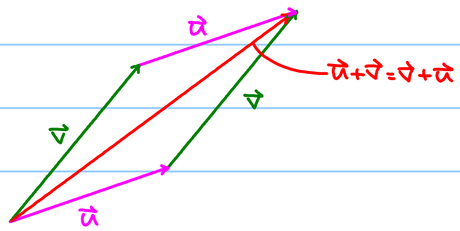
Remark: $\mathbb{R}^n$ is a vector space proof of (1):

Let $\vec{u}=(u_1,u_2,\cdots,u_n)$, $\vec{v}=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$.

$\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$

$=(v_1+u_1,v_2+u_2,\cdots,v_n+u_n)$ $\quad(\because u_i,v_i\in\mathbb{R},\ u_i+v_i=v_i+u_i)$

$=\vec{v}+\vec{u}$



Definition 1.6

If $\vec{v}=(v_1,v_2,\cdots,v_n)$, length of $\vec{v}$, $|\vec{v}|=\left(\sum_{i=1}^{n}v_i^2\right)^{\frac{1}{2}}=\sqrt{v_1^2+v_2^2+\cdots+v_n^2}$ (or denoted by $\|\vec{v}\|$).

Exercise 1.1

Let $\vec{v}\in\mathbb{R}^n$, $k\in\mathbb{R}$. Show that $|k\vec{v}|=|k||\vec{v}|$.

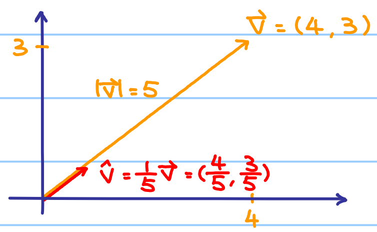
If we let $\hat{v}=\frac{1}{|\vec{v}|}\vec{v}$, then $\hat{v}\parallel\vec{v}$ and $|\hat{v}|=1$. $\hat{v}$ is said to be the unit vector of $\vec{v}$.

Idea: A vector $\vec{v}$ in $\mathbb{R}^n$ is a quantity with direction and magnitude.

$\vec{v}=|\vec{v}|\hat{v}$ where $\hat{v}$ and $|\vec{v}|$ give the direction and magnitude of $\vec{v}$ respectively.

Example 1.5

If $\vec{v}=(4,3)\in\mathbb{R}^2$, $|\vec{v}|=\sqrt{4^2+3^2}=5$ (Pyth thm.) and $\hat{v}=\frac{4}{5}\vec{i}+\frac{3}{5}\vec{j}$.



Definition 1.7 (Dot Product)

If $\vec{u}=(u_1,u_2,\cdots,u_n)$, $\vec{v}=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$, $\vec{u}\cdot\vec{v}=\sum_{i=1}^{n}u_iv_i=u_1v_1+u_2v_2+\cdots+u_nv_n$.

In particular, $\vec{v}\cdot\vec{v}=\sum_{i=1}^{n}v_i^2=|\vec{v}|^2$.

Example 1.5

If $\vec{u}=(4,2,3)$, $\vec{v}=(-1,6,-2)\in\mathbb{R}^3$, $\vec{u}\cdot\vec{v}=4\cdot(-1)+2\cdot6+3\cdot(-2)=2$.

Geometrical meaning ?

Cosine Law : $|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$

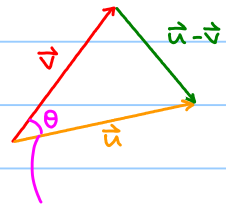
$$\sum_{i=1}^{n}(u_i - v_i)^2 = \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2 - 2|\vec{u}||\vec{v}|\cos\theta$$

$$\sum_{i=1}^{n} u_i^2 - 2\sum_{i=1}^{n} u_i v_i + \sum_{i=1}^{n} v_i^2 = \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2 - 2|\vec{u}||\vec{v}|\cos\theta$$

$$\sum_{i=1}^{n} u_i v_i = |\vec{u}||\vec{v}|\cos\theta$$

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

Triangle spanned by $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$.



Angle between $\vec{u}$ and $\vec{v}$

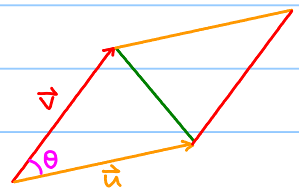
Direct consequence:

1) $\vec{u}$ is perpendicular (or orthogonal) to $\vec{v}$, i.e. $\vec{u}\perp\vec{v} \Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \vec{u}\cdot\vec{v} = 0$

2) $\vec{e_i}\cdot\vec{e_j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$

Furthermore, let $\vec{u} = u_1\vec{i} + u_2\vec{j}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} \in \mathbb{R}^2$.

The area of parallelogram spanned by $\vec{u}$ and $\vec{v}$



$= |\vec{u}||\vec{v}||\sin\theta|$

$= \sqrt{|\vec{u}|^2|\vec{v}|^2(1-\cos^2\theta)}$

$= \sqrt{|\vec{u}|^2|\vec{v}|^2 - |\vec{u}\cdot\vec{v}|^2}$

$= \sqrt{(u_1^2+u_2^2)(v_1^2+v_2^2) - (u_1v_1+u_2v_2)^2}$

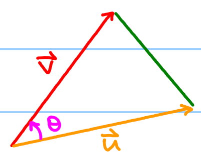
$= \sqrt{(u_1v_2 - u_2v_1)^2}$

$= |u_1v_2 - u_2v_1|$

Remark: Assume that θ is the angle measured from $\vec{u}$ to $\vec{v}$.

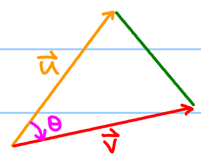
The signed area of parallelogram spanned by $\vec{u}$ and $\vec{v}$ $= |\vec{u}||\vec{v}|\sin\theta = u_1v_2 - u_2v_1 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

$u_1v_2 - u_2v_1 > 0$



$\theta > 0$

$u_1v_2 - u_2v_1 < 0$



$\theta < 0$

Proposition 1.2

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $t \in \mathbb{R}$.

1) (Commutative Law of Dot Product) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

2) (Distributive Law of Dot Product) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

3) $(t\vec{u})\cdot\vec{v} = \vec{u}\cdot(t\vec{v}) = t(\vec{u}\cdot\vec{v})$

proof of (2):

Let $\vec{u}=(u_1,u_2,\cdots,u_n)$, $\vec{v}=(v_1,v_2,\cdots,v_n)$ and $\vec{w}=(w_1,w_2,\cdots,w_n)$

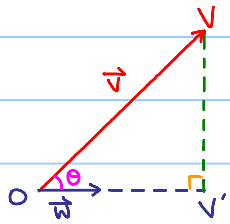
$\vec{u}\cdot(\vec{v}+\vec{w})=\sum_{i=1}^{n}u_i(v_i+w_i)=\sum_{i=1}^{n}(u_iv_i+u_iw_i)=\sum_{i=1}^{n}u_iv_i+\sum_{i=1}^{n}u_iw_i=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

Furthermore, $(\vec{v}+\vec{w})\,\vec{u}=\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}=\vec{v}\cdot\vec{u}+\vec{w}\cdot\vec{u}$.

Projection of $\vec{v}$ along $\vec{w}$:

length of $\overrightarrow{OV'}=|\vec{v}|\cos\theta$

$\text{proj}_{\vec{w}}(\vec{v})=\overrightarrow{OV'}=\underbrace{(|\vec{v}|\cos\theta)}_{\text{magnitude}}\underbrace{\hat{w}}_{\text{direction}}=\frac{|\vec{v}||\vec{w}|\cos\theta}{|\vec{w}|^2}\vec{w}=\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$



which is the projection of $\vec{v}$ along $\vec{w}$

$\overrightarrow{OV}=\vec{v}$ can be expressed as $\overrightarrow{OV'}+\overrightarrow{V'V}$

where $\overrightarrow{OV'}=\text{proj}_{\vec{w}}(\vec{v})=\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ and $\overrightarrow{V'V}=\overrightarrow{OV}-\overrightarrow{OV'}=\vec{v}-\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$

Furthermore, $\overrightarrow{OV'}\,/\!/\,\vec{w}$ and $\overrightarrow{V'V}\cdot\vec{w}=(\vec{v}-\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w})\cdot\vec{w}=0$, so $\overrightarrow{V'V}\perp\vec{w}$.

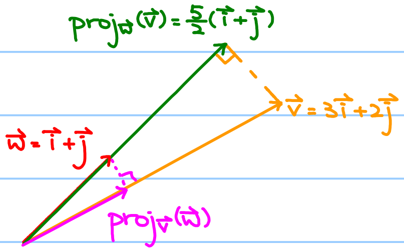
Example 1.6

Let $\vec{v}=3\vec{i}+2\vec{j}$, $\vec{w}=\vec{i}+\vec{j}\in\mathbb{R}^2$.

$\text{proj}_{\vec{w}}(\vec{v})=\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}=\frac{5}{2}(\vec{i}+\vec{j})$

Exercise: $\text{proj}_{\vec{v}}(\vec{w})=?$

Answer: $\text{proj}_{\vec{v}}(\vec{w})=\frac{5}{13}(3\vec{i}+2\vec{j})$



Example 1.7

Let $\vec{v}=2\vec{e}_1-3\vec{e}_2+\vec{e}_3+4\vec{e}_4$, $\vec{w}=\vec{e}_1+2\vec{e}_2-\vec{e}_3+\vec{e}_4\in\mathbb{R}^4$

$|\vec{v}|=\sqrt{2^2+(-3)^2+1^2+4^2}=\sqrt{30}$, $|\vec{w}|=\sqrt{1^2+2^2+(-1)^2+1^2}=\sqrt{7}$

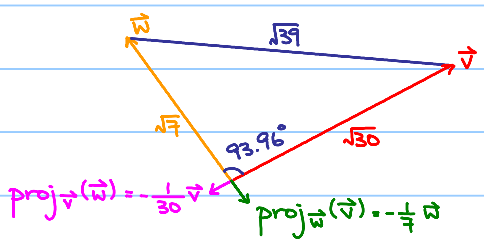
Distance between $\vec{v}$ and $\vec{w}$ $=|\vec{v}-\vec{w}|=|\vec{e}_1-5\vec{e}_2+2\vec{e}_3+3\vec{e}_4|=\sqrt{39}$

$\vec{v}\cdot\vec{w}=2\cdot1+(-3)\cdot2+1\cdot(-1)+4\cdot1=-1$

$|\vec{v}||\vec{w}|\cos\theta=\vec{v}\cdot\vec{w}$

$\sqrt{30}\,\sqrt{7}\cdot\cos\theta=-1$

$\theta=\cos^{-1}\left(\frac{-1}{\sqrt{210}}\right)\approx 93.96^\circ$



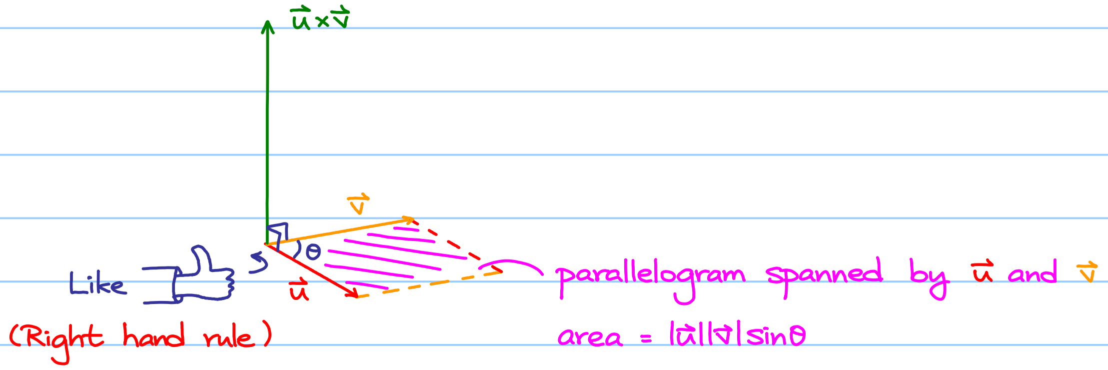
$\therefore$ Angle between $\vec{v}$ and $\vec{w}$ $\approx 93.96^\circ$

$\text{proj}_{\vec{w}}(\vec{v})=\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}=-\frac{1}{7}\vec{w}$

$\text{proj}_{\vec{v}}(\vec{w})=\frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}=-\frac{1}{30}\vec{v}$

Definition 1.8 (Cross Product in $\mathbb{R}^3$)

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$, $\vec{u}\times\vec{v}$ is defined as the following:



Caution: Cross product is only defined in $\mathbb{R}^3$ but NOT any other dimension.

Magnitude: $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

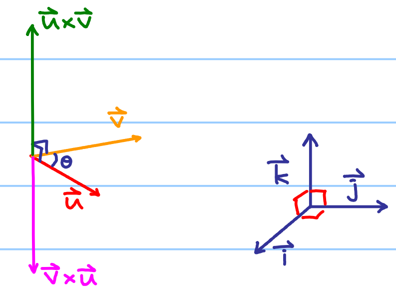
Direction: $\vec{u}\times\vec{v} \perp \vec{u}$ and $\vec{u}\times\vec{v} \perp \vec{v}$ with right hand rule.

By definition, we have:

1) $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

2) $\vec{i}\times\vec{j} = \vec{k}$, $\vec{j}\times\vec{k} = \vec{i}$, $\vec{k}\times\vec{i} = \vec{j}$

$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0}$ (NOT just the number 0)



How to compute $\vec{u}\times\vec{v}$ if $\vec{u} = u_1\vec{i}+u_2\vec{j}+u_3\vec{k}$ and $\vec{v} = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$?

$\vec{u}\times\vec{v} = (u_1\vec{i}+u_2\vec{j}+u_3\vec{k}) \times (v_1\vec{i}+v_2\vec{j}+v_3\vec{k})$ (Assume distributive law)

$$= u_1v_1\,\vec{i}\times\vec{i} + u_1v_2\,\vec{i}\times\vec{j} + u_1v_3\,\vec{i}\times\vec{k} +$$
$$u_2v_1\,\vec{j}\times\vec{i} + u_2v_2\,\vec{j}\times\vec{j} + u_2v_3\,\vec{j}\times\vec{k} +$$
$$u_3v_1\,\vec{k}\times\vec{i} + u_3v_2\,\vec{k}\times\vec{j} + u_3v_3\,\vec{k}\times\vec{k}$$

$$= (u_2v_3-u_3v_2)\vec{i} - (u_1v_3-u_3v_1)\vec{j} + (u_1v_2-u_2v_1)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

(You may forget all the above and take this as the definition of the cross product.)

Example 1.14

If $\vec{u} = \vec{i}+2\vec{k}$, $\vec{v} = 2\vec{i}-3\vec{j}+\vec{k}$,

then $\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ -3 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix}\vec{j} = 6\vec{i}+3\vec{j}-3\vec{k}$

Proposition 13

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, $t \in \mathbb{R}$.

1) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

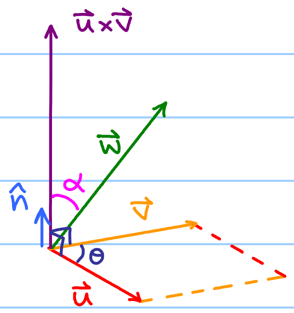
2) (Distributive Law of Cross Product) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

3) $(t\vec{u}) \times \vec{v} = \vec{u} \times (t\vec{v}) = t(\vec{u} \times \vec{v})$

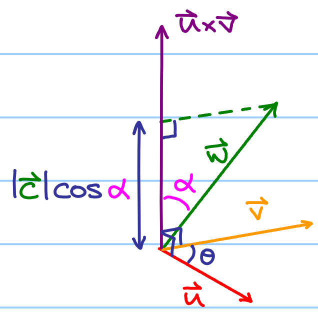
Note that if $\vec{u}, \vec{v} \in \mathbb{R}^3$, then $\vec{u} \times \vec{v} \in \mathbb{R}^3$.

Suppose $\vec{w} \in \mathbb{R}^3$, then we know that $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is well-defined and it is just a scalar.

$(\vec{u} \times \vec{v}) \cdot \vec{w}$ is called scalar triple product, but does it have any geometrical meaning?



$\hat{n}$: unit vector of $\vec{u} \times \vec{v}$

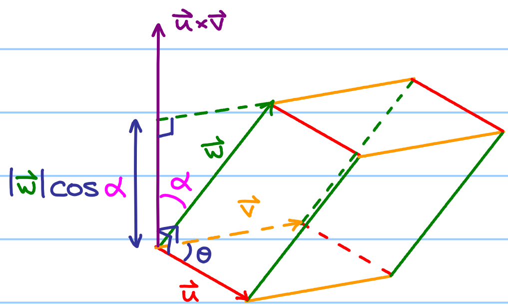


$\vec{u} \times \vec{v} = |\vec{u} \times \vec{v}| \, \hat{n}$

$(\vec{u} \times \vec{v}) \cdot \vec{w} = |\vec{u} \times \vec{v}| \, \hat{n} \cdot \vec{w}$

$$= \underbrace{|\vec{u} \times \vec{v}|}_{\text{base area}} \underbrace{(|\vec{w}| \cos \alpha)}_{\text{height}}$$

= (signed) volume of the parallelepiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$.



Remark: If $\frac{\pi}{2} < \alpha < \pi$, $\cos \alpha < 0$

If $\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$, $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$ and $\vec{w} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = [(u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}] \cdot (w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k})$$

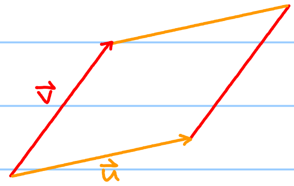
$$= (u_2 v_3 - u_3 v_2) w_1 - (u_1 v_3 - u_3 v_1) w_2 + (u_1 v_2 - u_2 v_1) w_3 = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

From the properties of determinants:

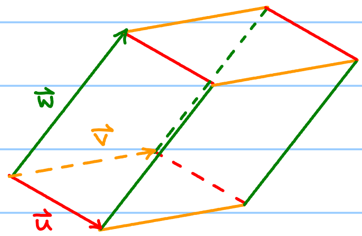
$$\left.\begin{array}{l} (\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u} = (\vec{w} \times \vec{u}) \cdot \vec{v} \\ (\vec{v} \times \vec{u}) \cdot \vec{w} = (\vec{u} \times \vec{w}) \cdot \vec{v} = (\vec{w} \times \vec{v}) \cdot \vec{u} \end{array}\right\} \text{differ by a minus sign.}$$

Think:

1) $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$ = signed area of parallelogram spanned by $\vec{u}=u_1\vec{i}+u_2\vec{j}$ and $\vec{v}=v_1\vec{i}+v_2\vec{j}$.



2) $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ = signed volume of the parallelepiped spanned by $\vec{u}=u_1\vec{i}+u_2\vec{j}+u_3\vec{k}$, $\vec{v}=v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$ and $\vec{w}=w_1\vec{i}+w_2\vec{j}+w_3\vec{k}$



3) If A is a n×n-real matrix, $|A|$ = ?

Two Inequalities

Proposition 14 (Cauchy Schwarz Inequality)

Let $a_1,a_2,\cdots,a_n,b_1,b_2,\cdots,b_n\in\mathbb{R}$. Then, $\left(\sum_{i=1}^n a_ib_i\right)^2\le\left(\sum_{i=1}^n a_i^2\right)\cdot\left(\sum_{i=1}^n b_i^2\right)$.

Furthermore, the equality holds if and only if $a_1=tb_1,\ a_2=tb_2,\cdots,a_n=tb_n$ for some $t\in\mathbb{R}$.

proof:

Let $\vec{a}=(a_1,a_2,\cdots,a_n)$, $\vec{b}=(b_1,b_2,\cdots,b_n)\in\mathbb{R}^n$ and let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta\le|\vec{a}||\vec{b}|$$

$$\therefore\ \left(\sum_{i=1}^n a_ib_i\right)^2\le\left(\sum_{i=1}^n a_i^2\right)\cdot\left(\sum_{i=1}^n b_i^2\right).$$

Furthermore, the equality holds $\Leftrightarrow$ $\theta=0$ or π $(\cos\theta=1)$, i.e. $\vec{a}\parallel\vec{b}$

$\Leftrightarrow$ $\vec{a}=t\vec{b}$, i.e. $a_1=tb_1,\ a_2=tb_2,\cdots,a_n=tb_n$, for some $t\in\mathbb{R}$.

Proposition 14 (Triangle Inequality)

Let A, B, C be three points in $\mathbb{R}^n$. Then, $AB+BC\ge AC$

Furthermore, the equality holds if and only if ΔABC is a degenerated triangle.

proof:

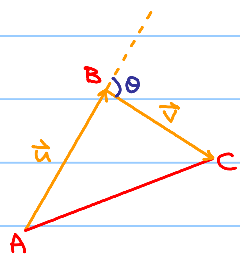
Let $\vec{u}=\overrightarrow{AB}$, $\vec{v}=\overrightarrow{BC}$. It is equivalent to show $|\vec{u}|+|\vec{v}|\ge|\vec{u}+\vec{v}|$.

$$|\vec{u}+\vec{v}|^2=(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$

$$=\vec{u}\cdot\vec{u}+2\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v}$$

$$\le|\vec{u}|^2+2|\vec{u}||\vec{v}|+|\vec{v}|^2 \qquad (\because\ \vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta\le|\vec{u}||\vec{v}|)$$

$$=(|\vec{u}|+|\vec{v}|)^2$$



Furthermore, the equality holds $\Leftrightarrow$ $\theta=0$ or π $(\cos\theta=1)$, i.e. $\vec{u}\parallel\vec{v}$

$\Leftrightarrow$ $AB\parallel BC$, i.e. ΔABC is a degenerated triangle.

§2 Straight Lines, Planes and Curves

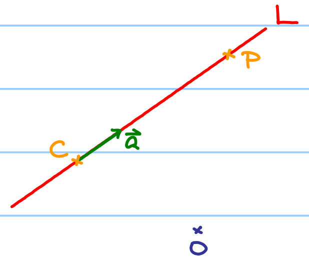
Straight line L in $\mathbb{R}^3$

Let $C=(c_1,c_2,c_3)$ be a fixed point

$P=(x,y,z)$ be a movable point

$\vec{a}=(a_1,a_2,a_3)$ be a fixed vector (direction vector)

L be a straight line passes through C and goes along direction $\vec{a}$.



Then, we have $\overrightarrow{CP} \parallel \vec{a}$, i.e $\overrightarrow{CP}=t\vec{a}$, $t\in\mathbb{R}$

$(x-c_1,y-c_2,z-c_3)=t(a_1,a_2,a_3)$

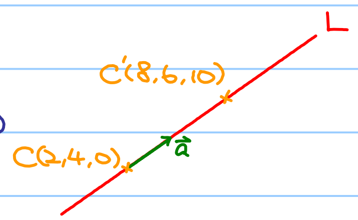
$$\therefore \begin{cases} x=c_1+ta_1 \\ y=c_2+ta_2 \\ z=c_3+ta_3 \end{cases} \quad \text{(parametric equation of } L\text{)}$$

Eliminate t: $\dfrac{x-c_1}{a_1}=\dfrac{y-c_2}{a_2}=\dfrac{z-c_3}{a_3}$ if $a_1,a_2,a_3\neq 0$.

(Think. If $a_1,a_2\neq 0$, but $a_3=0$, then the equation becomes: $\dfrac{x-c_1}{a_1}=\dfrac{y-c_2}{a_2}$ and $z=c_3$.)

Example 2.1

If the equation of a straight line L in $\mathbb{R}^3$ is $\dfrac{x-2}{3}=y-4=\dfrac{z}{5}$, then L passes through $(2,4,0)$ and goes along the direction $\vec{a}=(3,1,5)$

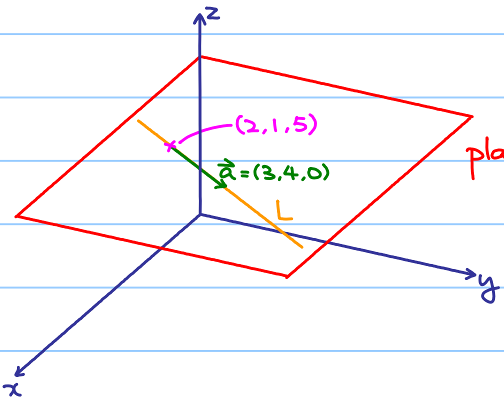


However, L also passes through the point $C'=(2,4,0)+2(3,1,5)=(8,6,10)$.

Therefore, $\dfrac{x-8}{3}=y-6=\dfrac{z-10}{5}$ is also an equation of L

Example 2.2

If the equation of a straight line L in $\mathbb{R}^3$ is $\dfrac{x-2}{3}=\dfrac{y-1}{4}$ and $z=5$, then L passes through $(2,1,5)$ and goes along the direction $\vec{a}=(3,4,0)$



plane π : $z=5$

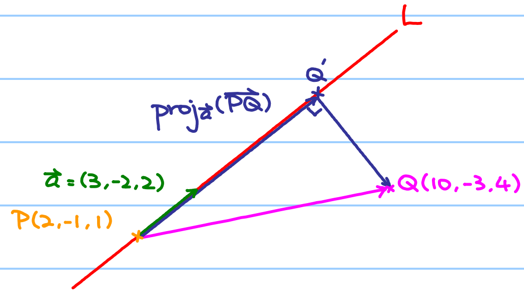
Note: L lies on π

Example 2.3

If L is a straight line in $\mathbb{R}^3$ given by the equation $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-1}{2}$,

$Q=(10,-3,4)$ is a fixed point.

What is the shortest distance between L and Q ?



L passes through $P(2,-1,1)$

Direction vector of $L = \vec{a} = (3,-2,2)$

$\overrightarrow{PQ} = (10,-3,4)-(2,-1,1) = (8,-2,3)$

$\overrightarrow{PQ'} = proj_{\vec{a}}(\overrightarrow{PQ}) = \frac{\overrightarrow{PQ}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} = \frac{34}{17}\vec{a} = 2\vec{a} = (6,-4,4)$

$\overrightarrow{Q'Q} = \overrightarrow{PQ} - \overrightarrow{PQ'} = (8,-2,3)-(6,-4,4) = (2,2,-1)$

Shortest distance between L and $Q = |\overrightarrow{Q'Q}| = \sqrt{2^2+2^2+(-1)^2} = 3$

Follow the idea of the discussion of straight lines in $\mathbb{R}^3$, figure out the equation of straight lines in $\mathbb{R}^n$

In general, if L is a straight line in $\mathbb{R}^n$ which passes through a fixed point $\vec{c}=(c_1,c_2,\cdots,c_n)$ and goes along the direction $\vec{a}=(a_1,a_2,\cdots,a_n)$.

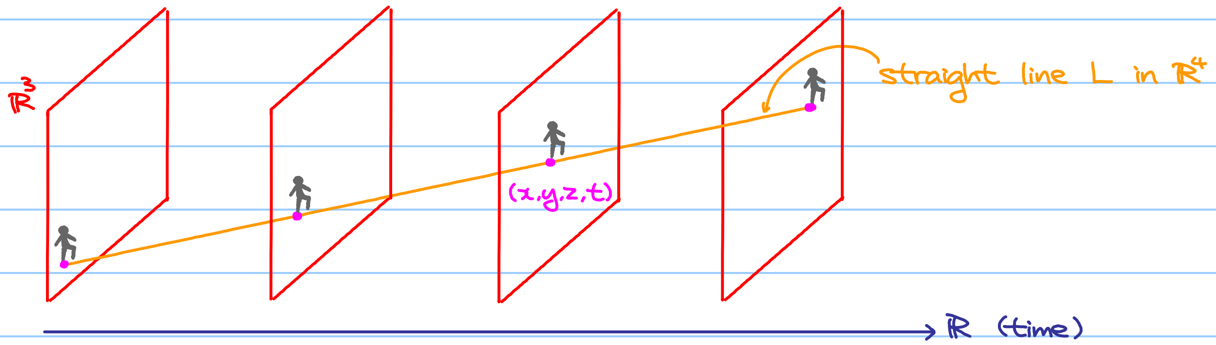
$\vec{x}=\vec{c}+t\vec{a}$, $t\in\mathbb{R}$ is a parametric equation of L, where $\vec{x}=(x_1,x_2,\cdots,x_n)$.

If $a_i \neq 0$ for all i, by eliminating t, we obtain $\frac{x_1-c_1}{a_1} = \frac{x_2-c_2}{a_2} = \cdots = \frac{x_n-c_n}{a_n}$.

(Think: What does the equation look like if some $a_i=0$?)

Need some imagination:

Somebody is walking in $\mathbb{R}^3$.

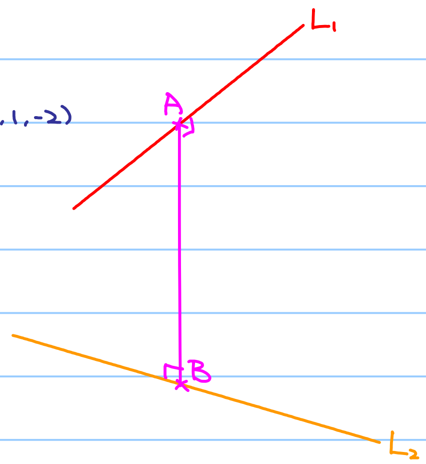


Example 2.4

Let $\vec{c_1}=(1,9,9,6)$, $\vec{a_1}=(2,-1,-3,2)$, $\vec{c_2}=(2,3,-2,7)$, $\vec{a_2}=(1,2,1,-2)$

Let $L_1: \vec{x}=\vec{c_1}+t\vec{a_1}$ and $L_2: \vec{x}=\vec{c_2}+s\vec{a_2}$, $t,s\in\mathbb{R}$, be two straight lines in $\mathbb{R}^4$.

Find the shortest distance between L_1 and L_2



Let $\vec{OA}=\vec{c_1}+t_0\vec{a_1}$, $\vec{OB}=\vec{c_2}+s_0\vec{a_2}$ for some $t_0,s_0\in\mathbb{R}$.

Then $\vec{BA}=\vec{OA}-\vec{OB}=(\vec{c_1}-\vec{c_2})+t_0\vec{a_1}-s_0\vec{a_2}=(-1,6,11,-1)+t_0(2,-1,-3,2)-s_0(1,2,1,-2)$

Note: $\vec{BA}\perp\vec{a_1}$ and $\vec{BA}\perp\vec{a_2}$ give two equations:

$$\begin{array}{lll} \vec{BA}\cdot\vec{a_1}=0 & \Rightarrow & -43+7s_0+18t_0=0 \\ \vec{BA}\cdot\vec{a_2}=0 & \Rightarrow & 24-10s_0-7t_0=0 \end{array} \Rightarrow \begin{cases} s_0=1 \\ t_0=2 \end{cases}$$

Idea: 2 equations, 2 unknowns, it suffices to know s_0 and t_0

$\therefore$ $\vec{AB}=(-1,6,11,-1)+2(2,-1,-3,2)-(1,2,1,-2)=(2,2,4,5)$

and the shortest distance between L_1 and $L_2=|\vec{AB}|=\sqrt{2^2+2^2+4^2+5^2}=\sqrt{49}=7$

Planes in $\mathbb{R}^3$

Let $Q=(q_1,q_2,q_3)$ be a fixed point on the plane.

$P=(x,y,z)$ be a movable point on the plane.

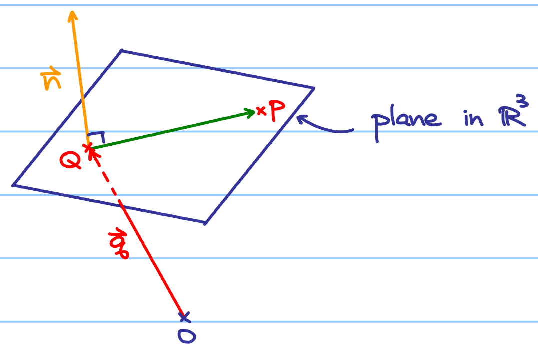
$\vec{n}=(A,B,C)$ be a normal of the plane.

Then, we have $\vec{n}\perp\vec{QP}$

i.e. $\vec{n}\cdot\vec{QP}=0$

$$A(x-q_1)+B(y-q_2)+C(z-q_3)=0$$

$$Ax+By+Cz+\underbrace{(-Aq_1-Bq_2-Cq_3)}_{\text{denote it by } D}=0$$



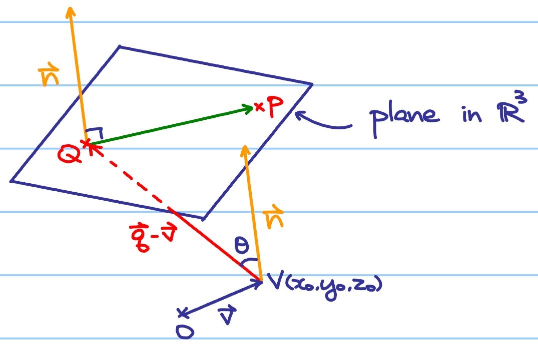
$\therefore$ The equation of a plane in $\mathbb{R}^3$ is of the form $Ax+By+Cz+D=0$

where $\vec{n}=A\vec{i}+B\vec{j}+C\vec{k}$ is a normal.

Furthermore, if d is the distance between $V(x_0,y_0,z_0)$ and the plane $\pi: Ax+By+Cz+D=0$ then $d=\left|\,|\vec{VQ}|\cos\theta\,\right|$ where θ is the angle between $\vec{n}$ and $\vec{VQ}$.

$$
\begin{aligned}
d &= \left|\,|\vec{VQ}|\cos\theta\,\right| \\
&= \left|\,|\vec{q}-\vec{v}|\cos\theta\,\right| \\
&= \left|\frac{|\vec{n}||\vec{q}-\vec{v}|\cos\theta}{|\vec{n}|}\right| \\
&= \frac{|\vec{n}\cdot(\vec{q}-\vec{v})|}{|\vec{n}|} \\
&= \frac{|(A,B,C)\cdot(q_1-x_0,\ q_2-y_0,\ q_3-z_0)|}{|\vec{n}|} \\
&= \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}
\end{aligned}
$$

(Recall : $D=-Aq_1-Bq_2-Cq_3$)



Example 2.5

$\pi: 2x-2y-z-3=0$ is a plane in $\mathbb{R}^3$ with a normal $2\vec{i}-2\vec{j}-\vec{k}$ in $\mathbb{R}^3$.

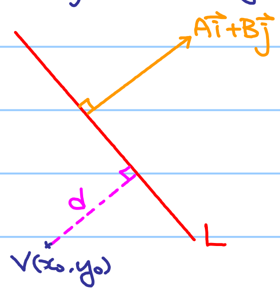
The distance between O and $\pi = \dfrac{|-3|}{\sqrt{2^2+(-2)^2+(-1)^2}} = 1$.

Exercise 2.1 (Revisit of straight lines in $\mathbb{R}^2$)

Follow the idea of the discussion of planes in $\mathbb{R}^3$, show that if $L: Ax+By+C=0$ is a straight line in $\mathbb{R}^2$, then

a) $\vec{n}=A\vec{i}+B\vec{j}$ gives a normal of L;

b) the distance between $V(x_0,y_0)$ and $L = d = \dfrac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$



Example 2.6

Let $L: \dfrac{x-1}{2}=\dfrac{y-2}{-1}=\dfrac{z}{2}$ be a straight line and $\pi: x+y+z=0$ be a plane in $\mathbb{R}^3$.

a) Find the intersection of L and π

b) Find the angle between L and π

c) Find the projection of L on π

a) If P is a point lying on L, $P=(1,2,0)+t(2,-1,2)=(1+2t,2-t,2t)$, $t\in\mathbb{R}$.

Suppose that P further lies on π, $(1+2t)+(2-t)+2t=0$

$$3t+3=0$$

$$t=-1$$

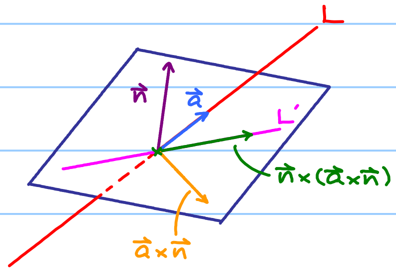
$\therefore$ L and π intersect at $(-1,3,-2)$.

b) Note: $\vec{a}=(2,-1,2)$ is a direction vector of L

$\vec{n}=(1,1,1)$ is a normal of π

The angle between L and $\vec{n}$ = $\cos^{-1}\left(\frac{\vec{a}\cdot\vec{n}}{|\vec{a}||\vec{n}|}\right)=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\therefore$ The angle between L and $\pi = \frac{\pi}{2}-\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$



c) Question: How to find a direction vector of L'?

$$\vec{a}\times\vec{n}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-1&2\\1&1&1\end{vmatrix}=-3\vec{i}+3\vec{k}$$

$$\vec{n}\times(\vec{a}\times\vec{n})=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&1&1\\-3&0&3\end{vmatrix}=3\vec{i}-6\vec{j}+3\vec{k}=3(\vec{i}-2\vec{j}+\vec{k})$$

$\therefore \vec{i}-2\vec{j}+\vec{k}$ is a direction vector of L'.

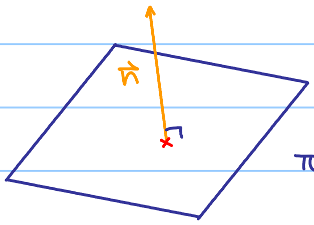
Equation of L': $x+1=\frac{y-3}{-2}=z+2$

Follow the idea of the discussion of planes in $\mathbb{R}^3$,

the equation $a_1x_1+a_2x_2+\cdots+a_nx_n+b=0$ in $\mathbb{R}^n$ gives a "plane" in $\mathbb{R}^n$,

which is said to be an affine hyperplane π

The vector $\vec{n}=(a_1,a_2,\cdots,a_n)$ is a normal of the affine hyperplane π.



π: $(n-1)$-dim affine hyperplane in $\mathbb{R}^n$, n-dim space.

1-dim affine hyperplane in $\mathbb{R}^2$ is just an usual straight line in $\mathbb{R}^2$.

2-dim affine hyperplane in $\mathbb{R}^3$ is just an usual plane in $\mathbb{R}^3$.

Example 2.7

Let Π in $\mathbb{R}^4$ given by $2x_1+x_2-x_3+3x_4=4$ and let $P=(1,2,3,1)$ be a point on Π.

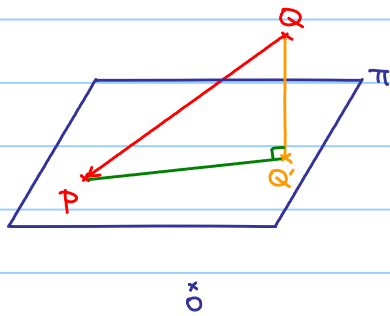
Also, let $Q=(2,5,7,4)$ be a point which does not lie on Π.

What is the projection Q' of Q on Π?

Note: $\vec{n}=(2,1,-1,3)$ is normal to Π, so

$$\overrightarrow{QQ'}=\mathrm{proj}_{\vec{n}}(\overrightarrow{QP})=\frac{\overrightarrow{QP}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}=\frac{-10}{15}(2,1,-1,3)=-\frac{2}{3}(2,1,-1,3)$$

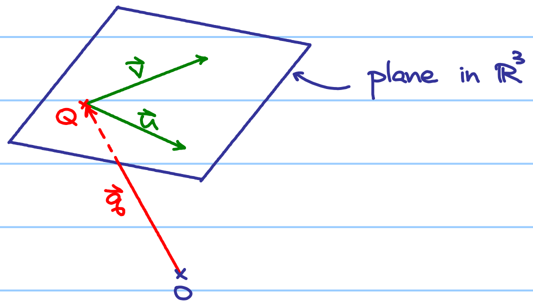
$$\therefore \overrightarrow{OQ'}=\overrightarrow{OQ}+\overrightarrow{QQ'}=\left(\frac{2}{3},\frac{13}{3},\frac{23}{3},2\right)$$



Parametric Equations.

Let $\vec{q} \in \mathbb{R}^3$, $\vec{u}, \vec{v} \in \mathbb{R}^3$ be two linearly independent (non-parallel) vectors.

$\vec{x} = \vec{q} + s\vec{u} + t\vec{v}$ is the parametric equation of the plane passing through $\vec{q}$ and containing $\vec{u}$ and $\vec{v}$.



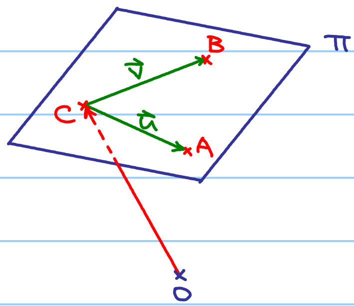
Example 2.7

Let $A=(1,0,0)$, $B=(2,1,1)$, $C=(-3,-2,-1)$ be three points in $\mathbb{R}^3$.

Then, $\overrightarrow{CA} = (4,2,1)$ and $\overrightarrow{CB} = (5,3,2)$.

$(x,y,z) = (-3,-2,-1) + s(4,2,1) + t(5,3,2)$

is a parametric equation of the plane π passing through A, B and C



$$\vec{n} = \overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & 1 \\ 5 & 3 & 2 \end{vmatrix} = \vec{i} - 3\vec{j} + 2\vec{k}$$

gives a normal of the plane π.

Let $P=(x,y,z)$ be a point on π. Then

$$\overrightarrow{CP} \cdot \vec{n} = 0$$

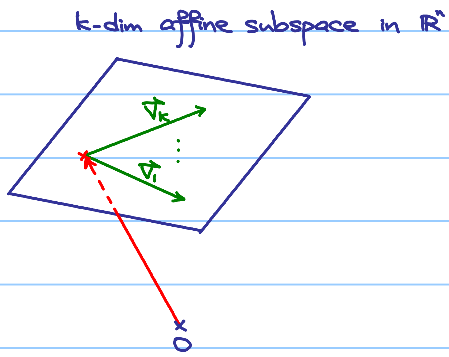
$$(x+3, y+2, z+1) \cdot (1,-3,2) = 0$$

$$x - 3y + 2z = 1$$

Follow the idea of the discussion of planes in $\mathbb{R}^3$,

Let $\vec{q} \in \mathbb{R}^3$, $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k \in \mathbb{R}^n$ be linearly independent vectors.

$\vec{x} = \vec{q} + t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k$ is the parametric equation of the k-dim affine subspace passing through $\vec{q}$ and containing $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$.



Curves

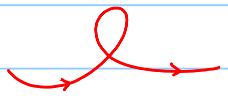
Definition 2.1

A (parametric) curve in $\mathbb{R}^n$ is a continuous function $\gamma : I \to \mathbb{R}^n$, where I is an interval

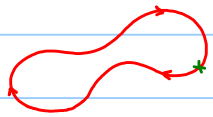
If we write the function as $\gamma(t) = (x_1(t), x_2(t), \dots, x_n(t))$, $t \in I$, that means each $x_i(t)$ is a continuous function.

Remark: $\gamma(t)$ is a vector in $\mathbb{R}^n$, so sometimes γ is called a vector function and some may write $\vec{\gamma}(t)$.

A curve $\gamma : I \to \mathbb{R}^n$ is said to be simple if γ is injective, i.e. if $\gamma(t_1) = \gamma(t_2)$, then $t_1 = t_2$



A curve $\gamma : [a,b] \to \mathbb{R}^n$ is said to be closed if $\gamma(a) = \gamma(b)$.



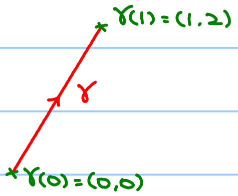
Example 2.8

Let $\gamma : [0,1] \to \mathbb{R}^2$ defined by $\gamma(t) = (x(t), y(t)) = (t, 2t)$.

Starting point: $\gamma(0) = (0,0)$

End point: $\gamma(1) = (1,2)$

$\begin{cases} x = t \\ y = 2t \end{cases} \xrightarrow{\text{Eliminate } t} y = 2x$



Definition 2.2

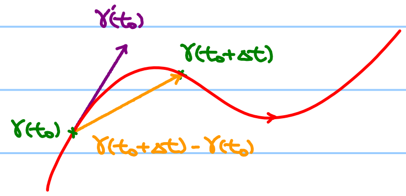
Let $\gamma : I \to \mathbb{R}^n$ be a curve and let $t_0 \in I$.

The derivative of γ at t_0 is defined as

$$\gamma'(t_0) = \lim_{\Delta t \to 0} \frac{\gamma(t_0 + \Delta t) - \gamma(t_0)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{x_1(t_0+\Delta t) - x_1(t_0)}{\Delta t}, \frac{x_2(t_0+\Delta t) - x_2(t_0)}{\Delta t}, \dots, \frac{x_n(t_0+\Delta t) - x_n(t_0)}{\Delta t} \right)$$

$$= (x_1'(t_0), x_2'(t_0), \dots, x_n'(t_0)) \quad \text{(if it exists)}$$



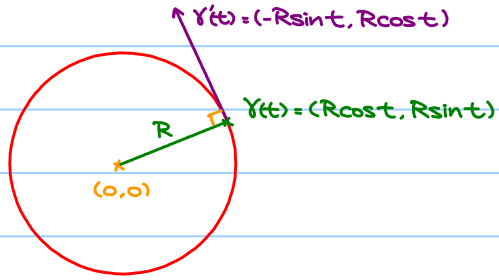
If $\gamma'(t)$ exists for all $t \in I$, then γ is said to be a differentiable curve

Remark: If t represents time and $\gamma(t)$ is the position of a moving particle, $\gamma'(t)$ is the velocity of that particle at time t. Hence, $|\gamma'(t)|$ is the speed of that particle at time t.

Example 2.9

Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ defined by

$\gamma(t) = (x(t), y(t)) = (R\cos t, R\sin t)$, where $R > 0$.



$$\begin{cases} x = R\cos t & -(1) \\ y = R\sin t & -(2) \end{cases}$$

$(1)^2 + (2)^2 : \quad x^2 + y^2 = R^2$

Therefore, γ is the circle centered at the origin with radius R.

$\gamma'(t) = (-R\sin t, R\cos t)$ and so $|\gamma'(t)| = \sqrt{(-R\sin t)^2 + (R\cos t)^2} = R$

Furthermore, let $\zeta : [0, \frac{2\pi}{w}] \to \mathbb{R}^2$ defined by $\zeta(t) = (x(t), y(t)) = (R\cos wt, R\sin wt)$, where $R, w > 0$.

Exercise: Check ζ also gives the same circle but $|\zeta'(t)| = Rw$.

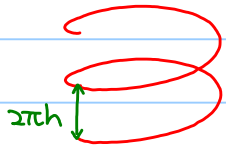
Therefore, different parametrizations may give the same curve

Exercise 2.2

Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by $\gamma(t) = (x(t), y(t), z(t)) = (R\cos t, R\sin t, ht)$, where $R, h > 0$.

What is γ?

Ans: Helix



Exercise 2.3

Give a parametrization of each of the following curves.

a) Circle given by $(x-h)^2 + (y-k)^2 = R^2$, where $h, k \in \mathbb{R}$, $R > 0$

b) Ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b > 0$

c) Line segment joining $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$

d) Curve given by graph of a continuous function $f : [a, b] \to \mathbb{R}$.

Ans: a) $\gamma(t) = (h + R\cos t, k + R\sin t)$, $t \in [0, 2\pi]$

b) $\gamma(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$

c) $\gamma(t) = (a_1, a_2, a_3) + t(b_1 - a_1, b_2 - a_2, b_3 - a_3)$, $t \in [0, 1]$

d) $\gamma(t) = (t, f(t))$, $t \in [a, b]$

Remark: It is more natural to use x as parameter and write $(x, f(x))$, $x \in [a, b]$

Proposition 2.1

Let $\gamma, \zeta : I \to \mathbb{R}^n$ be curves such that γ' and ζ' exist, let $f: I \to \mathbb{R}$ be a differentiable function, and let $c \in \mathbb{R}$. Then,

1) $(\vec{\gamma} \pm \vec{\zeta})'(t) = \vec{\gamma}'(t) \pm \vec{\zeta}'(t)$

2) $(c\vec{\gamma})'(t) = c\,\vec{\gamma}'(t)$

3) $(f \cdot \vec{\gamma})'(t) = f'(t)\,\vec{\gamma}(t) + f(t)\,\vec{\gamma}'(t)$

4) $(\vec{\gamma}(t) \cdot \vec{\zeta}(t))' = \vec{\gamma}'(t) \cdot \vec{\zeta}(t) + \vec{\gamma}(t) \cdot \vec{\zeta}'(t)$

5) If $n=3$, $(\vec{\gamma}(t) \times \vec{\zeta}(t))' = \vec{\gamma}'(t) \times \vec{\zeta}(t) + \vec{\gamma}(t) \times \vec{\zeta}'(t)$

proof:

4) Let $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t))$ and $\vec{\zeta}(t) = (\zeta_1(t), \zeta_2(t), \cdots, \zeta_n(t))$,

where $\gamma_i, \zeta_i : I \to \mathbb{R}$ are differentiable functions for $i = 1, 2, \cdots, n$

$$
\begin{aligned}
(\vec{\gamma}(t) \cdot \vec{\zeta}(t))' &= \frac{d}{dt}\left(\sum_{i=1}^{n} \gamma_i(t)\,\zeta_i(t)\right) \\
&= \sum_{i=1}^{n} \frac{d}{dt}(\gamma_i(t)\,\zeta_i(t)) \\
&= \sum_{i=1}^{n} (\gamma_i'(t)\,\zeta_i(t) + \gamma_i(t)\,\zeta_i'(t)) \\
&= \left(\sum_{i=1}^{n} \gamma_i'(t)\,\zeta_i(t)\right) + \left(\sum_{i=1}^{n} \gamma_i(t)\,\zeta_i'(t)\right) \\
&= \vec{\gamma}'(t) \cdot \vec{\zeta}(t) + \vec{\gamma}(t) \cdot \vec{\zeta}'(t)
\end{aligned}
$$

5) Let $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $\vec{\zeta}(t) = (\zeta_1(t), \zeta_2(t), \zeta_3(t))$,

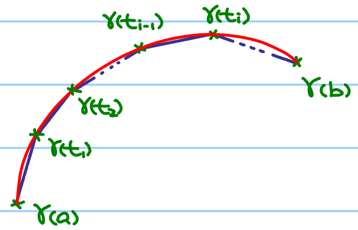
where $\gamma_i, \zeta_i : I \to \mathbb{R}$ are differentiable functions for $i = 1, 2, 3$.

$$\vec{\gamma}(t) \times \vec{\zeta}(t) = (\gamma_2(t)\zeta_3(t) - \gamma_3(t)\zeta_2(t))\,\vec{i} - (\gamma_1(t)\zeta_3(t) - \gamma_3(t)\zeta_1(t))\,\vec{j} + (\gamma_1(t)\zeta_2(t) - \gamma_2(t)\zeta_1(t))\,\vec{k}$$

$$
\begin{aligned}
(\vec{\gamma}(t) \times \vec{\zeta}(t))' = {} & \left[\frac{d}{dt}(\gamma_2(t)\zeta_3(t) - \gamma_3(t)\zeta_2(t))\right]\vec{i} - \left[\frac{d}{dt}(\gamma_1(t)\zeta_3(t) - \gamma_3(t)\zeta_1(t))\right]\vec{j} + {} \\
& \left[\frac{d}{dt}(\gamma_1(t)\zeta_2(t) - \gamma_2(t)\zeta_1(t))\right]\vec{k} \\
& \overset{\text{Ex.}}{=} \cdots \\
& = \vec{\gamma}'(t) \times \vec{\zeta}(t) + \vec{\gamma}(t) \times \vec{\zeta}'(t)
\end{aligned}
$$

Let $\gamma:[a,b]\to\mathbb{R}^n$ be a curve such that $\gamma'(t)$ exists.

Let $a=t_0<t_1<t_2<\cdots<t_n=b$ such that $t_i-t_{i-1}=\Delta t=\frac{b-a}{n}$.



Length of polygonal line $=\sum_{i=1}^{n}|\gamma(t_{i+1})-\gamma(t_i)|$

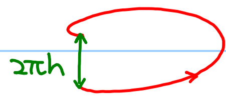
Taking limit ⇓

$$\begin{aligned}\text{Arclength of } \gamma &= \lim_{n\to\infty}\sum_{i=1}^{n}|\gamma(t_{i+1})-\gamma(t_i)| \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left|\frac{\gamma(t_{i+1})-\gamma(t_i)}{\Delta t}\right|\Delta t \\ &= \int_a^b|\gamma'(t)|\,dt \qquad \left(\text{As } n\to\infty,\ \Delta t=\frac{b-a}{n}\to 0\right)\end{aligned}$$

Example 2.10

Let $\gamma:[0,2\pi]\to\mathbb{R}^3$ defined by $\gamma(t)=(x(t),y(t),z(t))=(R\cos t, R\sin t, ht)$, where $R,h>0$.

$\gamma'(t)=(-R\sin t, R\cos t, h)$ and $|\gamma'(t)|=\sqrt{R^2+h^2}$



$$\begin{aligned}\text{Arclength of } \gamma &= \int_0^{2\pi}|\gamma'(t)|\,dt \\ &= \int_0^{2\pi}\sqrt{R^2+h^2}\,dt \\ &= 2\pi\sqrt{R^2+h^2}\end{aligned}$$

Example 2.11

Let $\gamma_1:[0,\pi]\to\mathbb{R}^2$ defined by $\gamma_1(t)=(x(t),y(t))=(\cos t,\sin t)$.

Let $\gamma_2:[-1,1]\to\mathbb{R}^2$ defined by $\gamma_2(t)=(x(t),y(t))=(-t,\sqrt{1-t^2})$.

Exercise: Show that both γ_1 and γ_2 are parametrizations of the upper semi-circle centered at the origin with radius 1.

$\gamma_1'(t)=(-\sin t,\cos t)$

$$\begin{aligned}\text{Arclength of } \gamma_1 &= \int_0^{\pi}|\gamma_1'(t)|\,dt \\ &= \int_0^{\pi}1\,dt \\ &= \pi\end{aligned}$$

$\gamma_2'(t)=\left(-1,\frac{-t}{\sqrt{1-t^2}}\right)$

$$\begin{aligned}\text{Arclength of } \gamma_2 &= \int_{-1}^{1}|\gamma_2'(t)|\,dt \\ &= \int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\cos\theta}\cos\theta\,d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1\,d\theta \\ &= \pi\end{aligned}$$

Let $t=\sin\theta$

$dt=\cos\theta\,d\theta$

When $t=-1$, $\theta=-\frac{\pi}{2}$;

$t=1$, $\theta=\frac{\pi}{2}$.

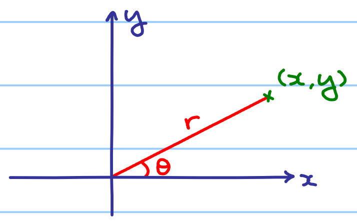
Exercise 2.4

Show that arclength of a curve is independent from choice of parametrization.

(Hint: Change of variables.)

§3 Polar, Cylinderical and Spherical Coordinates

Polar Coordinates



Change of coordinates:

$(r, \theta) \longrightarrow (x, y)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$(x, y) \longrightarrow (r, \theta)$

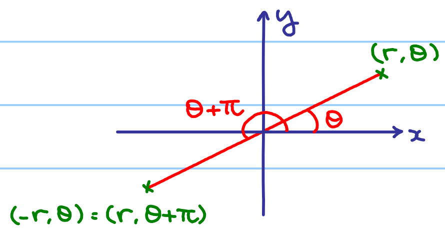
$$\begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$$

$(x, y) \neq (0, 0)$

$r, \theta \in \mathbb{R}$ but usually $r > 0$, $0 \leq \theta < 2\pi$

Convention: If $r = 0$, (r, θ) refers to the origin;

if $r < 0$, $(r, \theta) = (-r, \theta + \pi)$.



Exercise 3.1

Change of coordinates:

xy-coordinates	$\longleftrightarrow$	polar coordinates
a		$(6, \frac{\pi}{3})$
$(-2, 2)$		b
$(-2, -2\sqrt{3})$		c
d		$(-6, \frac{\pi}{3})$

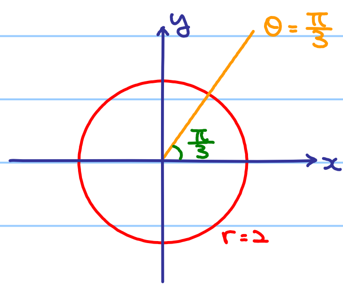
Hint: $= (6, \frac{4\pi}{3})$

Ans: a) $(3, 3\sqrt{3})$ b) $(2\sqrt{2}, \frac{3\pi}{4})$ c) $(4, \frac{4\pi}{3})$ d) $(-3, -3\sqrt{3})$

A polar equation is an equation on r and θ which defines an algebraic curve.

Example 3.1

In general, $r = r_0 > 0$ gives the circle centered at the origin with radius r_0; $r \geq 0$ and $\theta = \theta_0$ gives the ray originated from the origin where the angle swept from the positive x-axis to the ray (in anticlockwise direction) is θ_0.



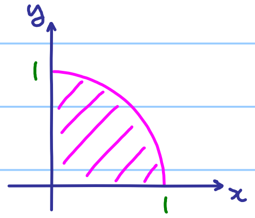
Exercise 3.2

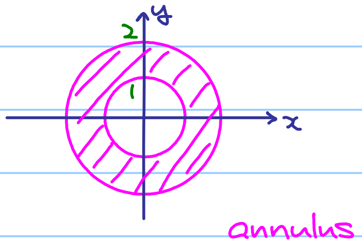
Draw the following subset of $\mathbb{R}^2$.

1) $D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\}$

2) $D = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta < 2\pi\}$.

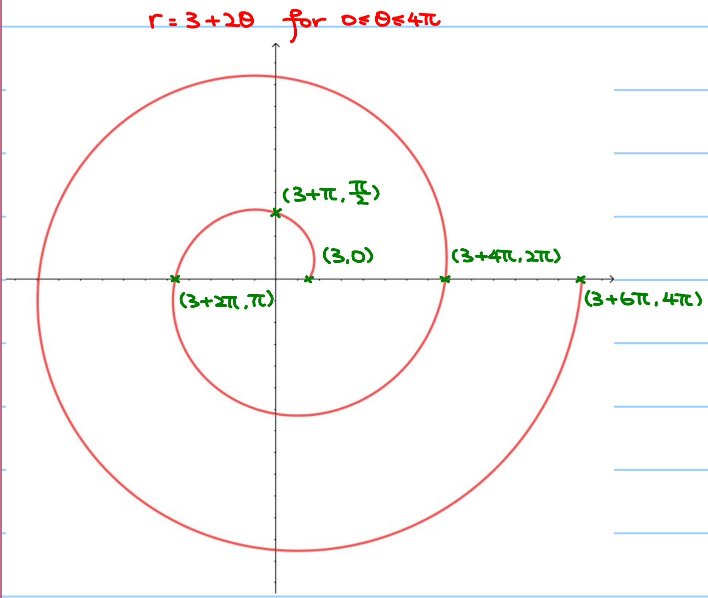
Ans:

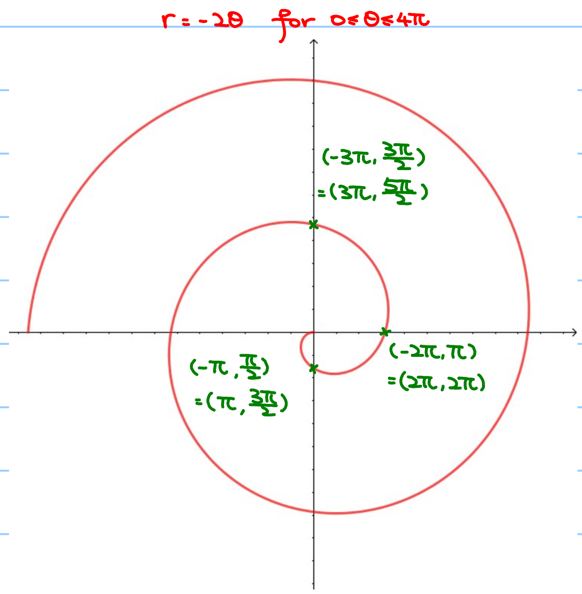
1) 

2) 

Example 3.2 (Archimedean Spiral)

$r = a + b\theta$, where $a, b \in \mathbb{R}$

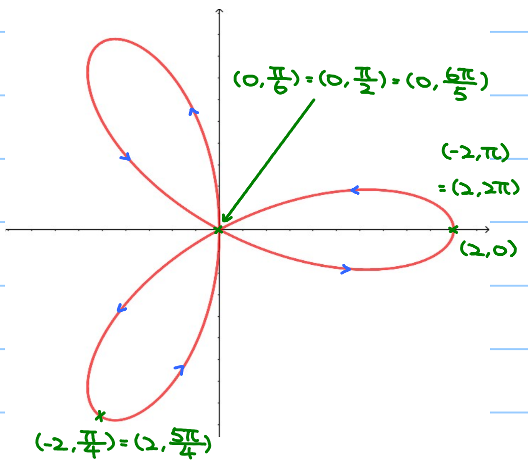




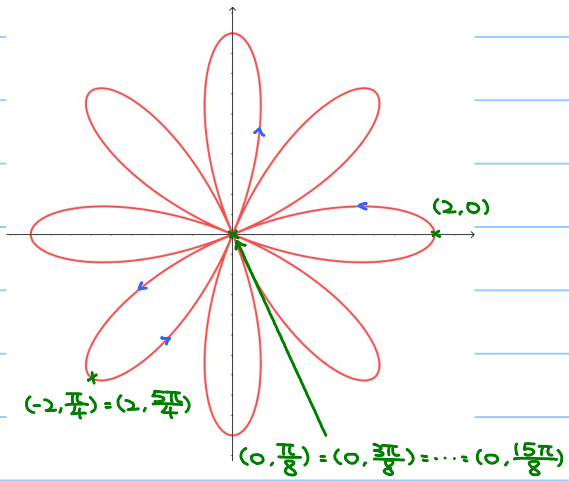
Example 3.3 (Polar Rose)

$r = a\cos(k\theta + \varphi)$, where $a, \varphi \in \mathbb{R}$, $k \in \mathbb{Z}^+$.

$r = 2\cos 3\theta$ for $0 \le \theta \le \pi$



$r = 2\cos 4\theta$ for $0 \le \theta \le 2\pi$



passing through the origin 8 times

In general, if k is odd, it has k petals ; if k is even, it has $2k$ petals.

Exercise 3.3

Draw the graphs of the following polar equations :

(i) (Cardioid) $r = 2a(1 - \cos\theta)$, where $a > 0$, $0 \le \theta \le 2\pi$;

(ii) (Limaçon) $r = b + a\cos\theta$, where $a, b \in \mathbb{R}$.

Suppose that $r = r(\theta)$, for $a \le \theta \le b$, defines a curve $\mathcal{C}$.

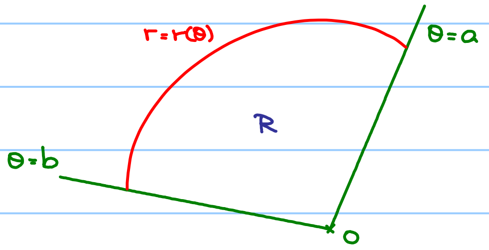
Question 1 : How do we find the arclength of $\mathcal{C}$?

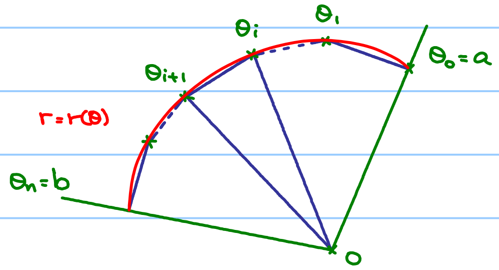
Note :
$$\begin{cases} x(\theta) = r(\theta)\cos\theta \\ y(\theta) = r(\theta)\sin\theta \end{cases} \Rightarrow \begin{cases} x'(\theta) = r'(\theta)\cos\theta - r(\theta)\sin\theta \\ y'(\theta) = r'(\theta)\sin\theta + r(\theta)\cos\theta \end{cases}$$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$\text{Arclength of } \mathcal{C} = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$
$$= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Question 2: What is the area of the region R bounded by $\theta = a$, $\theta = b$ and $r = r(\theta)$?





Area of i-th triangle $= \frac{1}{2} r_i \, r_{i+1} \sin(\theta_{i+1} - \theta_i) = \frac{1}{2} r_i \cdot r_{i+1} \sin\Delta\theta \approx \frac{1}{2} r_i^2 \Delta\theta$

Sum of areas of triangles $\approx \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta$ (Note $\lim_{\Delta\theta \to 0} \frac{\sin\Delta\theta}{\Delta\theta} = 1$, so $\sin\Delta\theta \approx \Delta\theta$)

Taking limit $\Downarrow$

Area of R $= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta$

$= \int_a^b \frac{1}{2} r^2 \, d\theta$

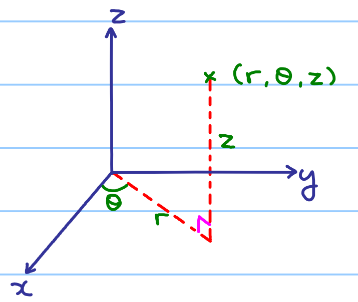
Exercise 3.4

Find the perimeter and area of a petal of $r = 2\cos 4\theta$.

Ans: perimeter $= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \sqrt{(2\cos 4\theta)^2 + (-8\sin 4\theta)^2} \, d\theta \approx 4.29$ area $= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{1}{2}(2\cos 4\theta)^2 d\theta = \frac{\pi}{4}$

Cylindrical Coordinates

Idea: Like polar coordinates but adding z-coordinates



Exercise 3.5

Change of coordinates:

xyz-coordinates $\longleftrightarrow$ cylindrical coordinates

a	$(8, \frac{\pi}{3}, 5)$
$(\sqrt{2}, -\sqrt{2}, -1)$	b
$(0, -3, 0)$	c

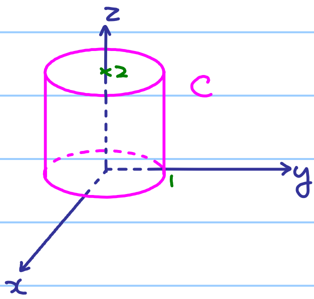
Ans: a) $(4, 4\sqrt{3}, 5)$ b) $(2, \frac{7\pi}{4}, -1)$ c) $(3, \frac{3\pi}{2}, 0)$

Example 3.4

Let $C = \{(x,y,z) : x^2+y^2 \le 1 \text{ and } 0 \le z \le 2\}$.

It can also be described by

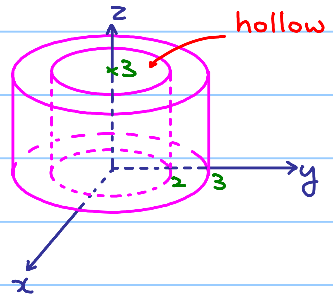
$C = \{(r,\theta,z) : 0 \le r \le 1,\ 0 \le \theta < 2\pi,\ 0 \le z \le 2\}$.



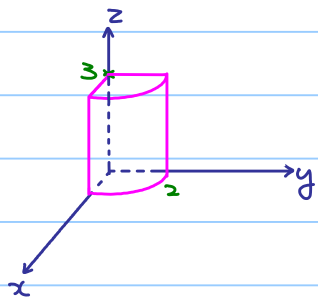
Exercise 3.6

Describe the following solid by cylindrical coordinates

1)



2)

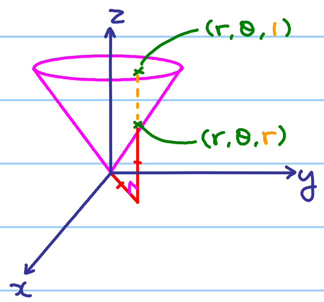


Ans : 1) $\{(r,\theta,z) : 2 \le r \le 3,\ 0 \le \theta < 2\pi,\ 0 \le z \le 3\}$

2) $\{(r,\theta,z) : 0 \le r \le 2,\ 0 \le \theta < \frac{\pi}{2},\ 0 \le z \le 3\}$

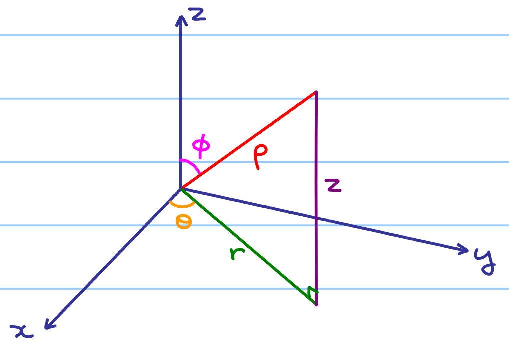
Example 3.5

$\{(r,\theta,z) : 0 \le r \le 1,\ 0 \le \theta < 2\pi,\ r \le z \le 1\}$



Spherical Coordinates

Describe a point in $\mathbb{R}^3$ by (ρ, ϕ, θ)



Note that: $r = \rho \sin\phi$

Change of coordinates:

$(\rho, \phi, \theta) \longrightarrow (x, y, z)$

$$\begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

$(x, y, z) \longrightarrow (\rho, \phi, \theta)$

$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z} \\ \tan\theta = \frac{y}{x} \end{cases}$$

(x,y,z) does not lie on z-axis

$\rho > 0$, $\phi, \theta \in \mathbb{R}$ but usually $0 < \phi < \pi$, $0 \le \theta < 2\pi$.

Convention: If $\phi = 0$ (or $\phi = \pi$), (ρ, ϕ, θ) refers to the point $(0, 0, \rho)$ (or $(0, 0, -\rho)$) in xyz-coordinates;

if $\rho = 0$, (ρ, ϕ, θ) refers to the origin.

Exercise 3.7

Change of coordinates:

xyz-coordinates	$\longleftrightarrow$	spherical coordinates
a		$(2, \frac{\pi}{4}, \frac{4\pi}{3})$
$(-1, 0, -\sqrt{3})$		b
$(0, -3, 0)$		c

Ans: a) $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \sqrt{2})$ b) $(2, \frac{5\pi}{6}, \pi)$ c) $(3, \frac{\pi}{2}, \frac{3\pi}{2})$

Example 3.6

$\{(\rho, \phi, \theta) : 0 \le \rho \le 2,\ 0 \le \phi \le \frac{\pi}{4},\ 0 \le \theta < 2\pi\}$